Prewavelet Solution to Poisson Equations

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Abstract

Finite element method is one of powerful numerical methods to solve PDE. Usually, if a finite element solution to a Poisson equation based on a triangulation of the underlying domain is not accurate enough, one will discard the solution and then refine the triangulation uniformly and compute a new finite element solution over the refined triangulation. It is wasteful to discard the original finite element solution. We propose a prewavelet method to save the original solution by adding a prewavelet subsolution to obtain the refined level finite element solution. To increase the accuracy of numerical solution to Poisson equations, we can keep adding prewavelet subsolutions.

Our prewavelets are orthogonal in the H^1 norm and they are compactly supported except for one globally supported basis function in a rectangular domain. We have implemented these prewavelet basis functions in MATLAB and used them for numerical solution of Poisson equation with Dirichlet boundary conditions. Numerical simulation demonstrates that our prewavelet solution is much more efficient than the standard finite element method.

1 Introduction

Finite element method is one of powerful numerical methods to solve PDE. Usually, if a finite element solution to a Poisson equation based on one level triangulation of the underlying domain is not accurate enough, one will discard the solution and then refine the triangulation and compute a new finite element solution at the refined level. It is wasteful to throw the original finite element solution away. In order to save the original solution and get the more

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accurate new solution, we have to add H^1 orthogonal subsolution. That is, let V_h be a finite element space over a triangulation Δ_h and $V_{h/2}$ be the finite element space over the refined triangulation. Since $V_h \subset V_{h/2}$, let $W_h = V_{h/2} \ominus V_h$ under H^1 norm, if $\Phi_h \in V_h$ is a finite element solution of Poisson equation with Dirichlet boundary condition, we can find $\Psi_h \in W_h$ so that $\Phi_h + \Psi_h$ is the finite element solution in $V_{h/2}$. In addition, suppose that ϕ_h is the most accurate solution that a computer can compute in the sense that it would be out of memory when computing a finite element solution $\Phi_{h/2}$ in $V_{h/2}$ directly. Since the size of the linear system associated with Ψ_h is smaller than $\Phi_{h/2}$, if the computer can solve Ψ_h, we can add Ψ_h to Φ_h to get $\Phi_{h/2}$ achieving the next level of accuracy. In this paper, we discuss how to compute Ψ_h. We shall construct compactly supported basis functions and a global supported basis function $\psi_{h,k}, k = 1, \cdots, N_h$ which span W_h. $\psi_{h,k}$'s are called prewavelets and Ψ_h is a linear combination of these $\psi_{h,k}$'s and hence is called a prewavelet subsolution.

Prewavelets have been studied for more than 10 years (cf. [9], [5]). There are many methods available to construct compactly supported prewavelets over 2D domains under the L_2 norm. That is $W_h = V_{h/2} \ominus V_h$ under L_2 norm, e.g., in a series of papers [6], [7], [8], [11], and [4]. In 1997, Bastin and Laubin ([2]) explained how to construct compactly supported orthonormal wavelets in Sobolev space in the univariate setting. See also [1] for biorthogonal wavelets in Sobolev space. In [14], Lorentz and Oswald showed that there is no compactly supported prewavelets in Sobolev space or under H^1 norm based on integer translations of a box spline over $\mathbf{R}^2$. Since continuous piecewise linear finite element can be expressed by using box spline B_{111}, the result in [14] ruins a hope to find compactly supported prewavelets under H^1 norm. But this is not an end of story. It is possible to construct compactly supported prewavelets in a semi-norm in the univariate setting in [10]. It is also possible to construct compactly supported prewavelets in H^r norm over each nested subspace, but the union of these prewavelets over all levels fails to be a stable basis for a Sobolev space (cf. [12]). Our new question is if we can find a prewavelet basis with as few as possible global supported prewavelet functions. Our anwser is affirmative. That is, there is a prewavelet basis for W_h with only one global supported basis function under the H^1 norm over rectangular domains. Also it is possible to find a compactly supported prewavelet basis for W_h under the H^1 norm for Poisson equation over a triangular domain (cf. [13]).

The paper is organized as follows: We first explain that the Dirichlet boundary value problem of Poisson equation can be converted into a Poisson equation with zero boundary condition. An explicit conversion will be given. Thus the H^1 norm is now equivalent to the H^1_0 semi-norm. Then we introduce some notation to explain the weak solution of Poisson equation and its approximation to the exact solution. These explanations are well-known and given in the Preliminary section §2. In §3, we explain how to construct compactly supported prewavelets under H^1_0 semi-norm. In §4, we explain how to implement our prewavelet method for numerical solution of Poisson equation. Finally in §5 we present some numerical results. Our numerical experiment show that the time for computing a finite element solution by our prewavelet method is about half of the time by the standard finite element method using the direct method for inverting the linear systems. If using the conjugate gradient method for the linear systems for the finite element method, the prewavelet method is still faster than

for sufficiently accurate iterative solutions.

2 Preliminary

Let us start with a square domain $\Omega = (0,1) \times (0,1) \in R^2$. Consider the Dirichlet boundary value problem for Poisson equation:

$$\begin{cases} -\Delta u(x,y) = g(x,y), & (x,y) \in \Omega \\ u(x,y) = f_1(x), & for \quad y=0 \quad and \quad 0 \le x \le 1 \\ u(x,y) = f_2(x), & for \quad y=1 \quad and \quad 0 \le x \le 1 \\ u(x,y) = f_3(y), & for \quad x=0 \quad and \quad 0 \le y \le 1 \\ u(x,y) = f_4(y), & for \quad x=1 \quad and \quad 0 \le y \le 1 \end{cases}$$

Without lose of generality, we may assume that $f_1(1) = f_2(1) = f_3(1) = f_4(1) = f_1(0) = f_2(0) = f_3(0) = f_4(0) = 0$. Otherwise, letting $f_1(0) = f_3(0) = a_1$, $f_3(1) = f_2(0) = a_2$, $f_2(1) = f_4(1) = a_3$, $f_4(0) = f_1(1) = a_4$, we define $h(x,y) = a_1 + (a_4 - a_1)x + (a_2 - a_1)y + (a_3 + a_1 - a_4 - a_2)xy$, and $v(x,y) = u(x,y) - h(x,y)$. Then the above Dirichlet problem becomes to:

$$\begin{cases} -\Delta v(x,y) = g(x,y), & (x,y) \in \Omega \\ v(x,y) = f_1(x) - h(x,0), & for \quad y=0 \quad and \quad 0 \le x \le 1 \\ v(x,y) = f_2(x) - h(x,1), & for \quad y=1 \quad and \quad 0 \le x \le 1 \\ v(x,y) = f_3(y) - h(0,y), & for \quad x=0 \quad and \quad 0 \le y \le 1 \\ v(x,y) = f_4(y) - h(1,y), & for \quad x=1 \quad and \quad 0 \le y \le 1 \end{cases}$$

which satisfy the above assumption.

Now let $w(x) = v(x,y) - x(f_4(y) - h(1,y)) - (1-x)(f_3(y) - h(0,y)) - y(f_2(x) - h(x,1)) - (1-y)(f_1(x) - h(x,0))$. Then $w(x)$ satisfies the equation

$$\begin{cases} -\Delta w(x,y) = g_1(x,y), & (x,y) \in \Omega \\ w(x,y) = 0, & (x,y) \in \partial\Omega \end{cases}$$

with $g_1(x,y) = g(x,y) + \frac{\partial^2}{\partial y^2}[-x(f_4(y) - h(1,y)) - (1-x)(f_3(y) - h(0,y))] + \frac{\partial^2}{\partial x^2}[-y(f_2(x) - h(x,1)) - (1-y)(f_1(x) - h(x,0))]$.

If we can find solution for w, it is easy to get $u(x,y)$. In the remaining paper, we only consider the Poisson equation with zero boundary condition:

$$\begin{cases} -\Delta u(x,y) = g(x,y), & (x,y) \in \Omega \\ u(x,y) = 0, & (x,y) \in \partial\Omega. \end{cases} \tag{1}$$

Next we define

$$H_0^1(\Omega) = \{v \in L^2(\Omega) : \quad \langle v, v \rangle_s < \infty \quad and \quad v(x,y) = 0, (x,y) \in \partial\Omega\},$$

where the inner product $\langle u, v\rangle_s$ is defined by

$$\langle u, v\rangle_s = \int_0^1 \int_0^1 \frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\frac{\partial v(x,y)}{\partial y} dxdy.$$

By using Poincare's inequality, $\|u\|_s = \sqrt{\langle u, u\rangle_s}$ is a standard Sobolev norm for $H_0^1(\Omega)$. Suppose $u, v \in H_0^1(\Omega)$. Integration by parts yields

$$\begin{aligned}\langle g, v\rangle &= \int_0^2 \int_0^2 g(x,y)v(x,y)dxdy \\ &= \int_0^2 \int_0^2 -\Delta u(x,y)v(x,y)dxdy \\ &= \int_0^2 \int_0^2 \frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\frac{\partial v(x,y)}{\partial y} dxdy \\ &= \langle u, v\rangle_s.\end{aligned}$$

Thus, a weak solution u to (1) is characterized by finding $u \in H_0^1(\Omega)$ such that

$$\langle u, v\rangle_s = \langle g, v\rangle, \quad \forall v \in H_0^1(\Omega). \tag{2}$$

The following result is well-known. For convenience, we present a short proof.

Theorem 2.1. *Suppose $g \in C(\Omega)$ and $u \in C^2(\Omega)$ satisfy (2). Then u is weak solution of (1).*

Proof. Let $v \in H_0^1(\Omega)$. Then integration by parts gives

$$\begin{aligned}\langle g, v\rangle &= \langle u, v\rangle_s \\ &= \int_0^1 \int_0^1 \frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\frac{\partial v(x,y)}{\partial y} dxdy \\ &= \int_0^1 \int_0^1 -\Delta u(x,y)v(x,y)dxdy \\ &= \langle -\Delta u(x,y), v\rangle.\end{aligned}$$

It follows that $\langle g - (-\Delta u(x,y)), v\rangle = 0$ for all $v \in H_0^1(\Omega)$. That is, $g \equiv -\Delta u$ and hence, u satisfies (1). $\square$

Next we introduce continuous linear spline space on $\Omega = [0,1] \times [0,1]$. For convenience, let $N_j = (2^j - 1)^2$ and $j \geq 1$. Denote $x_{ji} = \frac{i}{2^j} = y_{ji}$ for $i = 1, .., 2^j - 1$. Clearly, the lines segment of $x = x_{ji}$ and $y = y_{jk}$ divide the square Ω into N_j sub-squares. The diagonal going from down-left to up-right of each sub-square divides the sub-square into two congruent triangle. We will refer to the set of all such triangles as a Type-1 triangulation of Ω (see Figure 1).

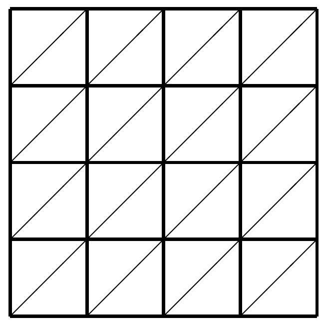

Figure 1. Type-I triangulation with j=2.

Define ϕ^j_{ik} to be linear spline with support on the hexagon with following vertices

$$\{(x_{j(i-1)}, y_{j(k-1)}), (x_{ji}, y_{j(k-1)}), (x_{j(i+1)}, y_{j(k)}), (x_{j(i+1)}, y_{j(k+1)}), (x_{j(i+1)}, y_{j(k)}), (x_{j(i-1)}, y_{j(k)})\}$$

and $\phi_{ik}(x_{ji'}, y_{jk'}) = \delta_{i,i'}\delta_{k,k'}$, where $\delta_{i,i'} = 0$ if $i' \neq i$ and 1 if $i' = i$.

Let $V_j = span\{\phi^j_{ik}, i = 1, .., 2^j - 1, k = 1, .., 2^j - 1\}$ be the subspace of $H^1_0(\Omega)$. By following lemma, there exists a unique $u_j \in V_j$ satisfying

$$\langle u_j, v\rangle_s = \langle f, v\rangle \ \ \forall v \in V_j. \tag{3}$$

u_j is the standard finite element solution in V_j. The following result is well-known. For completeness, we include a short proof.

Lemma 2.1. *Given $g \in L^2(\Omega)$, (3) has a unique solution.*

Proof. Reorder the basis functions $\phi^{(j)}_{ik}$ to ϕ_m, $m = 1, ..., N_j$ and let $u_j = \sum a_m\phi_m$. Denote $k_{mn} = \langle \phi_m, \phi_n\rangle_s$ and $F_m = \langle f, \phi_m\rangle$ for $m = 1, ..., N_j$. Set $A = (a_m)$ to be the coefficient vector, $K = [k_{mn}]_{1\le m,n\le N_j}$ to be the stiff matrix, and $F = (F_m)$ to be the right hand side vector. Then the solutions in (3) is written in the following matrix equation form

$$KA = F. \tag{4}$$

We claim that the solution for above equation always exists and is unique. Otherwise there is a nonzero vector $\mathbf{c}$ such that $K\mathbf{c} = 0$. Write $\mathbf{c} = (c_m, m = 1, ..., N_j)$ and let $v = \sum_{i=1}^{N_j} c_i\phi_i$ be the linear spline. Then $K\mathbf{c} = 0$ is equivalent to

$$\langle v, \phi_m\rangle_s = 0 \quad \forall m = 1, \cdots, N_j.$$

Multiplying $\langle v, \phi_m\rangle_s$ by c_m and summing over m yields $\langle v, v\rangle_s = 0$. Thus, $v = a + bx + cdxy$. Boundary condition implies $v \equiv 0$. Since $\{\phi_m\}$ are linear independent, $\mathbf{c} \equiv 0$ and hence, the solution is unique. $\square$

Let us discuss the error between u and u_j. It is standard in finite element analysis (cf. [3]). For completeness we present a simple derivation. Subtracting (3) from (2) implies

$$\langle u - u_j, w\rangle_s = 0 \qquad \forall w \in V_j. \tag{5}$$

Then for any $v \in V_j$

$$\begin{aligned} \|u-u_j\|_s^2 &= \langle u-u_j, u-u_j\rangle_s \\ &= \langle u-u_j, u-v\rangle_s + \langle u-u_j, v-u_j\rangle_s \\ &= \langle u-u_j, u-v\rangle_s \\ &\le \|u-u_j\|_s \|u-v\|_s \end{aligned}$$

It follows that $\|u-u_j\|_s \le \|u-v\|_s$ for any $v \in V_j$. Thus we have proved the following.

Lemma 2.2. *(Céa's Lemma)* $\|u-u_j\|_s = \min\{\|u-v\|_s : v \in V_j\}$.

Given $u \in C^0(\Omega)$, let $u_j \in V_j$ be the interpolant of v:

$$u_j = \sum_{ik} u(x_{ji}, y_{jk})\phi_{ik}^{(j)}.$$

The following error estimate is well-known.

Lemma 2.3. *Suppose* $u \in C^2(\Omega)$. *Then*

$$\|u-u_j\|_s \le \frac{\sqrt{12}}{2^j}\sqrt{\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^\infty}^2 + \left\|\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right\|_{L^\infty}^2 + \left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^\infty}^2}.$$

Proof. The proof is elementary and is left to the reader. See [13] for detail. □

3 Multiresolution and Prewavelets over Type-I triangulations

We start with the definition of multi-resolution approximation of $H_0^1(\Omega)$:

Definition 3.1. *A multiresolution approximation of* $H_0^1(\Omega)$ *is a sequence of finite dimensions subspaces* V_j, $j \in Z^+$ *of* $H_0^1(\Omega)$ *such that*

(1) $V_j \subset V_{j+1}, \quad j \in Z^+$;

(2) $\bigcup_{j=1}^\infty V_j$ *is dense in* $H_0^1(\Omega)$.

Let Γ^j be the type-1 triangulation with $2N_j$ triangles. Naturally, let Γ^{j+1} be the uniform refinement of Γ^j. Let V_j be the continuous piecewise linear spline space defined on the previous section. That is, $V_j = span\{\phi^j_{ik}, i = 1, .., 2^j - 1, k = 1, .., 2^j - 1\}$, where ϕ^j_{ik} are continuous piecewise linear functions which is 1 at (x_{ji}, y_{jk}) and zero at all other vertices. Let $V_{j+1} = span\{\phi^{j+1}_{ik}, i = 1, .., 2^{j+1}-1, k = 1, .., 2^{j+1}-1\}$, and $(x_{j+1,i}, y_{j+1,k})$ are the vertices on the j+1 level Type-1 triangulation. Then the refinement equation is easily seen to be

$$\phi^j_{ik} = \phi^{j+1}_{2i,2k} + \frac{1}{2}\phi^{j+1}_{2i-1,2k} + \frac{1}{2}\phi^{j+1}_{2i-1,2k-1} + \frac{1}{2}\phi^{j+1}_{2i,2k-1} + \frac{1}{2}\phi^{j+1}_{2i+1,2k} + \frac{1}{2}\phi^{j+1}_{2i+1,2k+1} + \frac{1}{2}\phi^{j+1}_{2i,2k+1}.$$

See the Figure 2.

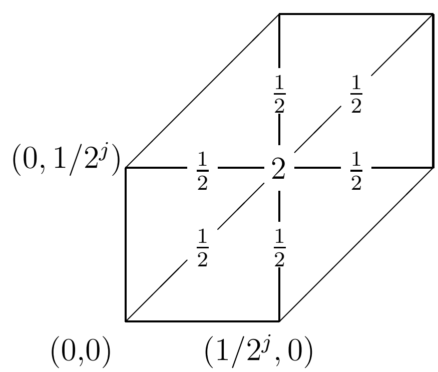

Figure 2. Dilation relations

The main purpose of this paper is to build a basis for the orthogonal complement W_j of V_j in V_{j+1} under the inner product $\langle \cdot, \cdot \rangle_s$. Suppose we have the W_j. Then $V_{j+1} = V_j + W_j$ under the $H^1_0(\Omega)$ inner product. For a solution u_j satisfying (3), we do not have to find out the solution for

$$u_{j+1} \in V_{j+1} \text{ such that } \langle u_{j+1}, v \rangle_s = \langle g, v \rangle \;\; \forall v \in V_{j+1}.$$

Instead, we only need to find solutions for

$$w_j \in W_j \text{ such that } \langle w_j, v \rangle_s = \langle g, v \rangle \;\; \forall v \in W_j.$$

Then we have $w_j + u_j = u_{j+1}$. Ideally, we hope the supports of basis functions for W_j are small, since small support can accelerate the calculations of $\langle g, v \rangle_s$. As explained in the Introduction, there is no compactly supported prewavelets for W_j. Neverthless, we shall construct basis functions with only one globally supported basis function for W_j in the following.

Clearly the Γ_j can be continuously refined and hence we will have a nested sequence of subspaces

$$V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5 ......$$

to span $H^1_0(\Omega)$ by Lemmma 2.3 since $C^2(\Omega)$ is dense in $H^1_0(\Omega)$.

Let $W_j \subset V_{j+1}$ be the orthogonal complement of V_j in V_{j+1} for each refinement level j, i.e.,

$$V_{j+1} = V_j \bigoplus W_j.$$

Then we get the decomposition

$$V_{j+1} = V_1 \bigoplus W_1 \bigoplus W_2 \bigoplus W_3 \bigoplus ...... \bigoplus W_j$$

for any $j \geq 1$. The weak solution u_{j+1} to the Poisson equation (1) at V_{j+1} can be built by

$$u_{j+1} = u_1 + w_1 + w_2 + \cdots + w_j.$$

We now focus on building basis functions for the orthogonal complement W_j. By direct calculation, we obtain the following lemma immediately.

Lemma 3.1. *We have* $\langle \phi^j_{ik}, \phi^{j+1}_{2i,2k}, \rangle_s = 2$,
$\langle \phi^j_{ik}, \phi^{j+1}_{2i-1,2k}, \rangle_s = 1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i,2k-1}, \rangle_s = 1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i+1,2k}, \rangle_s = 1/2,$
$\langle \phi^j_{ik}, \phi^{j+1}_{2i,2k+1}, \rangle_s = 1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i-1,2k-1}, \rangle_s = 1, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i+1,2k+1}, \rangle_s = 1,$
$\langle \phi^j_{ik}, \phi^{j+1}_{2i-2,2k}, \rangle_s = -1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i+2,2k}, \rangle_s = -1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i,2k-2}, \rangle_s = -1/2,$
$\langle \phi^j_{ik}, \phi^{j+1}_{2i,2k+2}, \rangle_s = -1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i-2,2k-2}, \rangle_s = 0, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i+2,2k+2}, \rangle_s = 0,$
$\langle \phi^j_{ik}, \phi^{j+1}_{2i-2,2k-1}, \rangle_s = -1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i-1,2k+1}, \rangle_s = -1, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i+1,2k+2}, \rangle_s = -1/2,$
$\langle \phi^j_{ik}, \phi^{j+1}_{2i+2,2k+1}, \rangle_s = -1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i+1,2k-1}, \rangle_s = -1, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i-1,2k-2}, \rangle_s = -1/2,$
$\langle \phi^j_{ik}, \phi^{j+1}_{i',k'}, \rangle_s = 0$, *for other* i', k' *which are not listed above.*

Let ψ^j be a function in W_j. Since $W_j \subset V_{j+1}$, let us write $\psi^j = \sum_{ik} \phi^{j+1}_{ik} b_{ik}$ for some unknown coefficients b_{ik}. Then by orthogonal condition $\langle \phi^j_{i'k'}, \psi^j \rangle_s = 0$, we need to solve the following equations.

$$0 = \langle \phi^j_{i'k'}, \sum_{i,k} b_{ik} \phi^{j+1}_{ik} \rangle_s = \sum_{i,k} b_{ik} \langle \phi^j_{i'k'}, \phi^{j+1}_{ik} \rangle_s. \tag{6}$$

Each $(i'k')$ determines one equation. Since there are N_j elements in the set V_j, they determine the N_j equations. These N_j equations with N_{j+1} coefficients, $b_{i,k}$. There are at least $N_{j+1} - N_j$ degrees of freedom. The solution space of these equation system should be the W_j. The linear independence of $\phi^j_{i',k'}$ implies that the coefficient matrix of the above linear system is of full rank. Hence, there are $N_{j+1} - N_j$ linear independent solutions which constitute a basis for W_j.

Definition 3.2. *Let* $V^m_{j+1} = span\{\phi^{j+1}_{ik}, i = 1, .., 2m-1, k = 1, .., 2m-1\}$ *be a subspace of* V_{j+1}. *Let* W^m_j *be subspace of* W_j *such that* $W^m_j = W_j \bigcap V^m_{j+1}$.

Obviously $\emptyset \subset V^1_{j+1} \subset V^2_{j+1} \subset \ldots \subset V^{2^j}_{j+1} = V_{j+1}$, and $\emptyset \subset W^1_j \subset W^2_j \subset \ldots \subset W^{2^j}_j = W_j$. There is no nonzero solution of (6) in space of V^1_{j+1}. However, there are five solution of (6) in space V^2_{j+1}. They are solutions of the following system of linear equations.

$$\sum_{1\le i,k\le 3} b_{ik}\left\langle \phi_{ik}^{j+1},\phi_{1,1}^{j}\right\rangle_s=0,\quad \sum_{1\le i,k\le 3} b_{ik}\left\langle \phi_{ik}^{j+1},\phi_{2,1}^{j}\right\rangle_s=0,$$
$$\sum_{1\le i,k\le 3} b_{ik}\left\langle \phi_{ik}^{j+1},\phi_{1,2}^{j}\right\rangle_s=0,\quad \sum_{1\le i,k\le 3} b_{ik}\left\langle \phi_{ik}^{j+1},\phi_{2,2}^{j}\right\rangle_s=0.$$

They are equivalent to the following equations.

$$\begin{pmatrix} \langle\phi_{1,1}^{j},\phi_{1,1}^{j+1}\rangle_s & \langle\phi_{1,1}^{j},\phi_{2,1}^{j+1}\rangle_s & \cdots\cdot & \langle\phi_{1,1}^{j},\phi_{3,3}^{j+1}\rangle_s \\ \langle\phi_{2,1}^{j},\phi_{1,1}^{j+1}\rangle_s & \langle\phi_{2,1}^{j},\phi_{2,1}^{j+1}\rangle_s & \cdots\cdot & \langle\phi_{2,1}^{j},\phi_{3,3}^{j+1}\rangle_s \\ \langle\phi_{1,2}^{j},\phi_{1,1}^{j+1}\rangle_s & \langle\phi_{1,2}^{j},\phi_{2,1}^{j+1}\rangle_s & \cdots\cdot & \langle\phi_{1,2}^{j},\phi_{3,3}^{j+1}\rangle_s \\ \langle\phi_{2,2}^{j},\phi_{1,1}^{j+1}\rangle_s & \langle\phi_{2,2}^{j},\phi_{2,1}^{j+1}\rangle_s & \cdots\cdot & \langle\phi_{2,2}^{j},\phi_{3,3}^{j+1}\rangle_s \end{pmatrix}\begin{pmatrix} b_{1,1}\\ b_{2,1}\\ b_{3,1}\\ b_{1,2}\\ b_{2,2}\\ b_{3,2}\\ b_{1,3}\\ b_{2,3}\\ b_{3,3}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}.$$

Using Lemma 3.1, we obtain the following equations.

$$\begin{pmatrix} 1 & 1/2 & -1 & 1/2 & 2 & 1/2 & -1 & 1/2 & 1\\ 0 & -1/2 & 1 & 0 & -1/2 & 1/2 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & -1/2 & 1\\ 0 & 0 & 0 & -1/2 & -1/2 & 0 & 1 & 1/2 & -1 \end{pmatrix}\begin{pmatrix} b_{1,1}\\ b_{2,1}\\ b_{3,1}\\ b_{1,2}\\ b_{2,2}\\ b_{3,2}\\ b_{1,3}\\ b_{2,3}\\ b_{3,3}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}.$$

The rank of the left matrix is four, because $\phi_{1,1}^{j}$, $\phi_{2,1}^{j}$, $\phi_{1,2}^{j}$, $\phi_{2,2}^{j}$, are linear independent. So there are five solutions as shown below.

$$\begin{pmatrix} b_{1,1}\\ b_{2,1}\\ b_{3,1}\\ b_{1,2}\\ b_{2,2}\\ b_{3,2}\\ b_{1,3}\\ b_{2,3}\\ b_{3,3}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\2\\0\\0\\1\\0\\0\end{pmatrix}\text{ or }\begin{pmatrix}0\\2\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}\text{ or }\begin{pmatrix}1\\1\\0\\1\\-1\\0\\0\\0\\0\end{pmatrix}\text{ or }\begin{pmatrix}0\\0\\0\\0\\-1\\1\\0\\1\\1\end{pmatrix}\text{ or }\begin{pmatrix}0\\-1\\0\\1\\0\\-1\\0\\1\\0\end{pmatrix}.$$

More precisely,

$$\psi^{j,1}_{0,1} = 2\phi^{j+1}_{1,2} + \phi^{j+1}_{1,3} \quad \text{as shown in Figure 3;} \tag{7}$$

$$\psi^{j,2}_{1,0} = 2\phi^{j+1}_{2,1} + \phi^{j+1}_{3,1} \quad \text{as shown in Figure 4;} \tag{8}$$

$$\psi^{j,3}_{1,1} = -\phi^{j+1}_{2,2} + \phi^{j+1}_{3,2} + \phi^{j+1}_{2,3} + \phi^{j+1}_{3,3} \quad \text{as shown in Figure 5;} \tag{9}$$

$$\psi^{j,4}_{1,1} = \phi^{j+1}_{1,1} + \phi^{j+1}_{2,1} + \phi^{j+1}_{1,2} - \phi^{j+1}_{2,2} \quad \text{as shown in Figure 6;} \tag{10}$$

$$\psi^{j,5}_{1,1} = \phi^{j+1}_{1,2} + \phi^{j+1}_{2,3} - \phi^{j+1}_{2,1} - \phi^{j+1}_{3,2} \quad \text{as shown in Figure 7.} \tag{11}$$

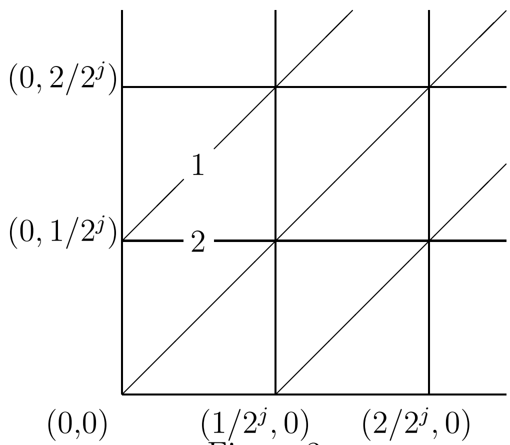

Figure 3.

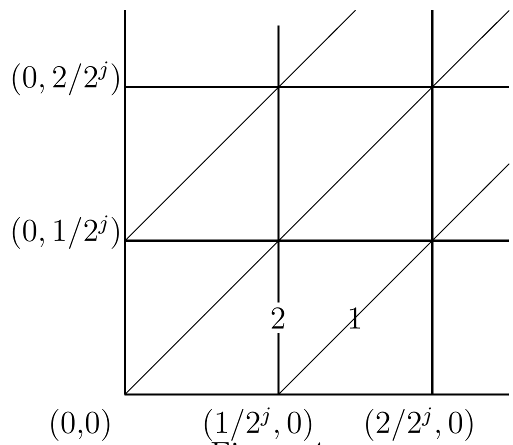

Figure 4.

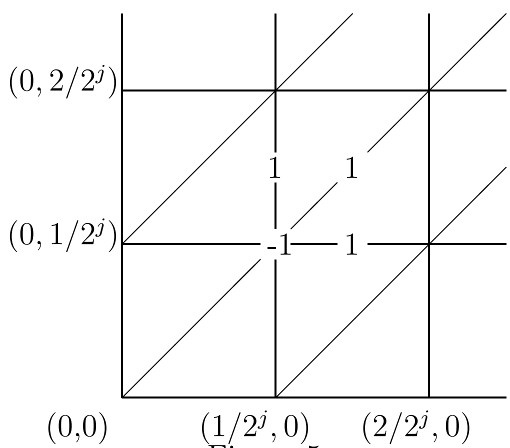

Figure 5.

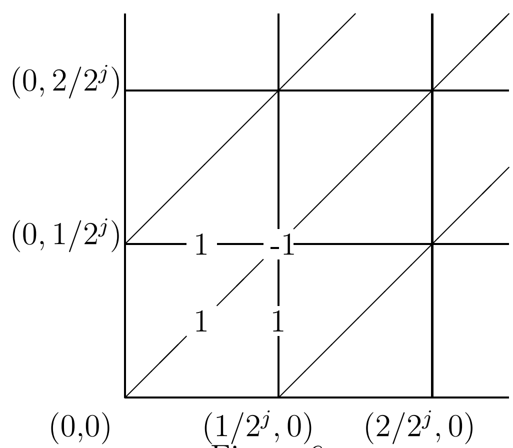

Figure 6.

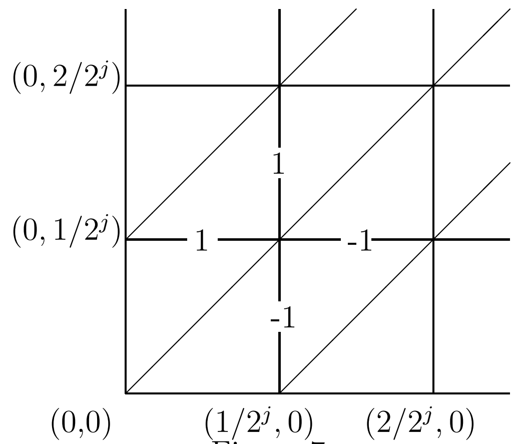

Figure 7.

Now we consider V_j^3. Similarly, there are 25 non-zero coefficient for linear system (6) and the coefficient matrix of rank 9. So the dimension of solution space of W_j^3 is $25-9=16$. The first five of them are the same to the wavelet functions in (7)–(11). The other 11 are given below.

$$
\begin{aligned}
\psi_{0,2}^{j,1} &= 2\phi_{1,4}^{j+1} + \phi_{1,5}^{j+1} && \text{as shown in Figure 8;}\\
\psi_{2,0}^{j,2} &= 2\phi_{4,1}^{j+1} + \phi_{5,1}^{j+1} && \text{as shown in Figure 9;}\\
\psi_{1,2}^{j,3} &= \phi_{3,5}^{j+1} + \phi_{3,4}^{j+1} + \phi_{2,5}^{j+1} - \phi_{2,4}^{j+1} && \text{as shown in Figure 10;}\\
\psi_{2,2}^{j,3} &= \phi_{5,5}^{j+1} + \phi_{5,4}^{j+1} + \phi_{4,5}^{j+1} - \phi_{4,4}^{j+1} && \text{as shown in Figure 11;}\\
\psi_{2,1}^{j,3} &= \phi_{5,3}^{j+1} + \phi_{5,2}^{j+1} + \phi_{4,3}^{j+1} - \phi_{4,2}^{j+1} && \text{as shown in Figure 12;}\\
\psi_{2,1}^{j,4} &= \phi_{3,2}^{j+1} + \phi_{4,1}^{j+1} + \phi_{3,1}^{j+1} - \phi_{4,2}^{j+1} && \text{as shown in Figure 13;}\\
\psi_{2,2}^{j,4} &= \phi_{3,3}^{j+1} + \phi_{4,3}^{j+1} + \phi_{3,4}^{j+1} - \phi_{4,4}^{j+1} && \text{as shown in Figure 14;}\\
\psi_{1,2}^{j,4} &= \phi_{1,3}^{j+1} + \phi_{2,3}^{j+1} + \phi_{1,4}^{j+1} - \phi_{2,4}^{j+1} && \text{as shown in Figure 15;}\\
\psi_{1,2}^{j,5} &= \phi_{1,4}^{j+1} + \phi_{2,5}^{j+1} - \phi_{2,3}^{j+1} - \phi_{3,4}^{j+1} && \text{as shown in Figure 16;}\\
\psi_{2,2}^{j,5} &= \phi_{3,4}^{j+1} + \phi_{4,5}^{j+1} - \phi_{4,3}^{j+1} - \phi_{5,4}^{j+1} && \text{as shown in Figure 17;}\\
\psi_{2,1}^{j,5} &= \phi_{3,2}^{j+1} + \phi_{4,3}^{j+1} - \phi_{4,1}^{j+1} - \phi_{5,2}^{j+1} && \text{as shown in Figure 18.}
\end{aligned}
$$

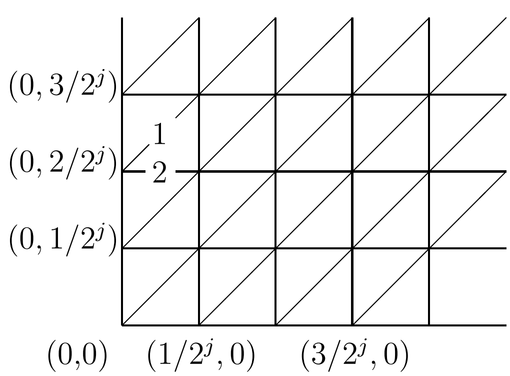

Figure 8.

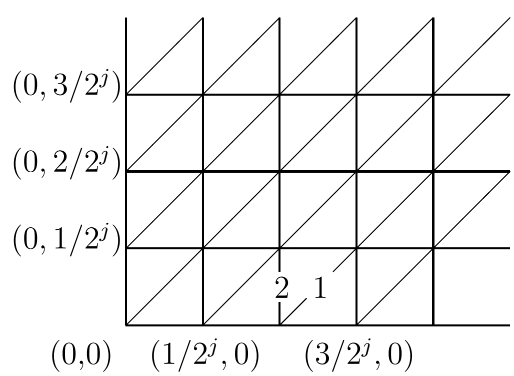

Figure 9.

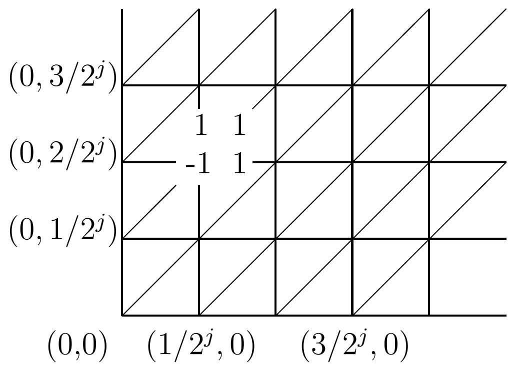

Figure 10.

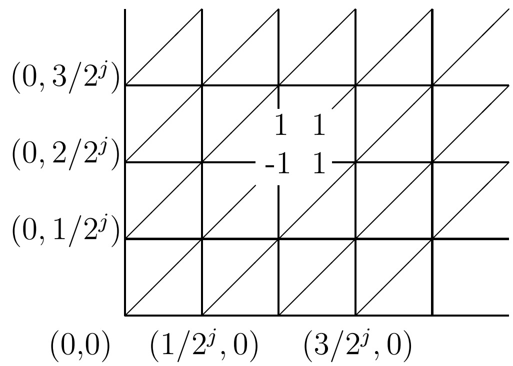

Figure 11.

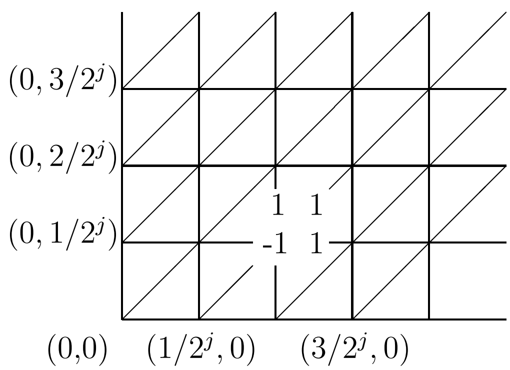

Figure 12.

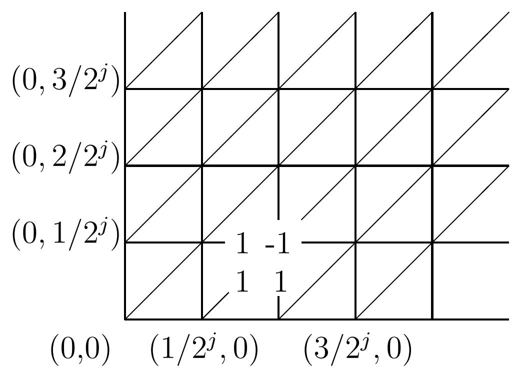

Figure 13.

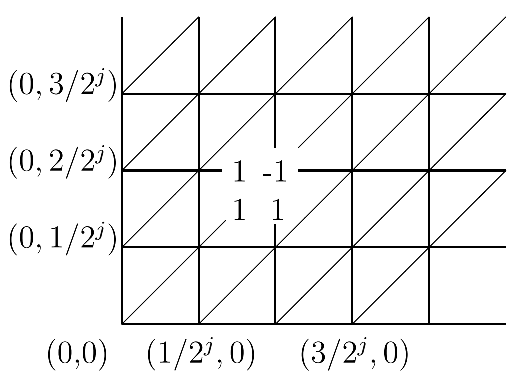

Figure 14.

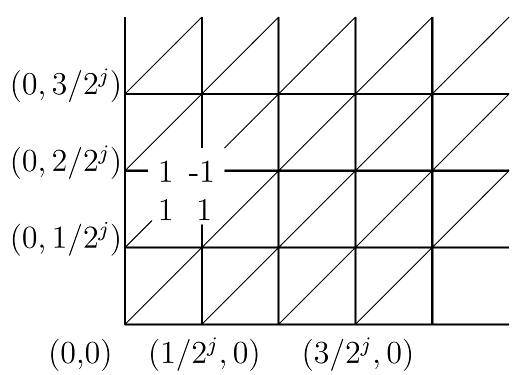

Figure 15.

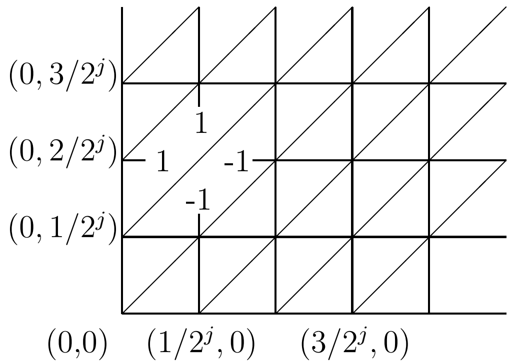

Figure 16.

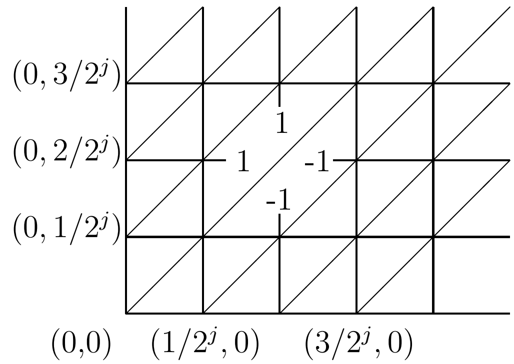

Figure 17.

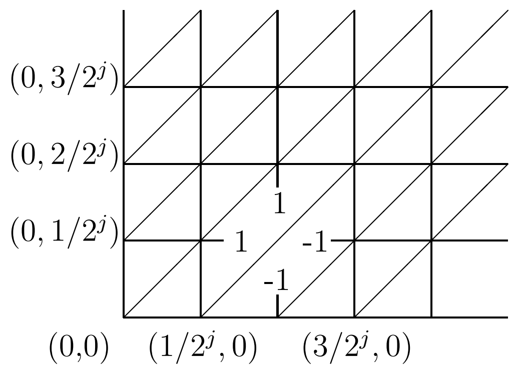

Figure 18.

The above computation can be carried out on V_j^n for $n = 3, ...2^j - 1$. We have thus obtained five types of wavelet functions:

$$\psi_{0,k}^{j,1} = 2\phi_{1,k+1}^{j+1} + \phi_{1,k+2}^{j+1}$$

is supported next to the vertical boundary and is called vertical boundary wavelet.

$$\psi_{k,0}^{j,2} = 2\phi_{k+1,1}^{j+1} + \phi_{k+2,1}^{j+1}$$

called horizontal boundary wavelet, is supported next to the horizontal boundary. The next three types are supported inside the domain. The following

$$\psi_{i,k}^{j,3} = -\phi_{i+1,k+1}^{j+1} + \phi_{i+2,k+1}^{j+1} + \phi_{i+1,k+2}^{j+1} + \phi_{i+2,k+2}^{j+1}$$

is called interior wavelet of first kind. We call

$$\psi_{i,k}^{j,4} = -\phi_{2i,2k}^{j+1} + \phi_{2i-1,2k}^{j+1} + \phi_{2i,2k-1}^{j+1} + \phi_{2i-1,2k-1}^{j+1}$$

interior wavelet of second kind. The last one

$$\psi_{i,k}^{j,5} = \phi_{2i-1,2k}^{j+1} + \phi_{2i,2k+1}^{j+1} - \phi_{2i,2k-1}^{j+1} - \phi_{2i+1,2k}^{j+1}$$

is called interior wavelet of third kind.

Theorem 3.1. *All the five types of wavelets in the V_{j+1}^n are linear independent for $1 \le n \le 2j-1$. That is, for each $1 \le n \le 2j-1$, the following functions*

$$\begin{array}{ll} \psi_{0,k}^{j,1}, & k = 1,..,n-1, \\ \psi_{k,0}^{j,2}, & k = 1,..,n-1, \\ \psi_{i,k}^{j,3}, & 1 \le i,k \le n-1, \\ \psi_{i,k}^{j,4}, & 1 \le i,k \le n-1, \\ \psi_{i,k}^{j,5}, & 1 \le i,k \le n-1 \end{array}$$

are linear independent.

Proof. Let us prove it by induction. It is true for $n = 2$ and for $n = 3$. Suppose it is true for $n = p$, that is,

$$\begin{array}{ll} \psi_{0,k}^{j,1}, & k = 1,..,p-1; \\ \psi_{k,0}^{j,2}, & k = 1,..,p-1; \\ \psi_{i,k}^{j,3}, & 1 \le i,k \le p-1; \\ \psi_{i,k}^{j,4}, & 1 \le i,k \le p-1; \\ \psi_{i,k}^{j,5}, & 1 \le i,k \le p-1; \end{array}$$

are linear independent. For $n = p+1$, there are $6p-1$ new functions which are

$$\begin{array}{ll} \psi_{0,k}^{j,1}, & k = p; \\ \psi_{k,0}^{j,2}, & k = p; \\ \psi_{i,k}^{j,3}, & i \quad or \quad k = p; \\ \psi_{i,k}^{j,4}, & i \quad or \quad k = p; \\ \psi_{i,k}^{j,5}, & i \quad or \quad k = p. \end{array}$$

Suppose they are not linear independent. That is, one can find

$$\begin{array}{ll} a^1, & \\ a^2, & \\ a_{i,k}^3, & i \quad or \quad k = p; \\ a_{i,k}^4, & i \quad or \quad k = p; \\ a_{i,k}^5, & i \quad or \quad k = p \end{array}$$

such that

$$a^1\psi_{0,p}^{j,1} + a^2\psi_{p,0}^{j,2} + \sum_{i \text{ or } k=p} a_{i,k}^3\psi_{i,k}^{j,3} + \sum_{i \text{ or } k=p} a_{i,k}^4\psi_{i,k}^{j,4} + \sum_{i \text{ or } k=p} a_{i,k}^5\psi_{i,k}^{j,5} + \psi' = 0, \tag{12}$$

where ψ' is linear combination of the following functions:

$$\begin{array}{ll} \psi_{0,k}^{j,1}, & k = 1,..,p-1; \\ \psi_{k,0}^{j,2}, & k = 1,..,p-1; \\ \psi_{i,k}^{j,3}, & 1 \le i,k \le p-1; \\ \psi_{i,k}^{j,4}, & 1 \le i,k \le p-1; \\ \psi_{i,k}^{j,5}, & 1 \le i,k \le p-1. \end{array}$$

By the definition, $\phi^{j+1}_{2i+1,2k+1}, i = p$ or $k = p$ appear only once in $\psi^{j,3}_{i,k}, i = p$ or $k = p$, $\psi^{j,1}_{0,p}$ and $\psi^{j,2}_{p,0}$. Since ϕ^{j+1} are linear independent, that is, $a^3_{i,k} = 0, i \quad or \quad k = p$, $a^1 = 0$, and $a^2 = 0$. Thus the equation (12) can be simplified to

$$\sum_{i\, or\, k=p} a^4_{i,k}\psi^{j,4}_{i,k} + \sum_{i\, or\, k=p} a^5_{i,k}\psi^{j,5}_{i,k} + \psi^{'} = 0. \tag{13}$$

By the similar reason, $\phi^{j+1}_{2i,2k}, i = p$ or $k = p$ appear only once in $\psi^{j,4}_{i,k}, i = p$ or $k = p$. Since ϕ^{j+1}_{ik} are linear independent, $a^4_{i,k} = 0, i\, or\, k = p$. Thus the equation (13) can be further simplified to the following equation

$$\sum_{i\, or\, k=p} a^5_{i,k}\psi^{j,5}_{i,k} + \psi^{'} = 0.$$

Similarly, $a^5_{i,k} = 0, i\, or\, k = p$ too. Thus the equation (12) is reduced to

$$\psi^{'} = 0.$$

By induction hypothesis, all the coefficient of $\psi^{'} = 0$ are zeros. Hence,

$$\begin{array}{ll} \psi^{j,1}_{0,k}, & k = 1, .., n-1, \\ \psi^{j,2}_{k,0}, & k = 1, .., n-1, \\ \psi^{j,3}_{i,k}, & 1 \le i, k \le n-1, \\ \psi^{j,4}_{i,k}, & 1 \le i, k \le n-1, \\ \psi^{j,5}_{i,k}, & 1 \le i, k \le n-1 \end{array}$$

are linear independent. □

Theorem 3.2. *All the five types of wavelets in the W^n_j form a basis of W^n_j for $1 \le n \le 2j-1$. That is,*

$$W^n_j = span\{\psi^{j,1}_{0,k}, \psi^{j,2}_{k,0}, \psi^{j,3}_{i,k}, \psi^{j,4}_{i,k}, \psi^{j,5}_{i,k}, 1 \le i, k \le n-1\}$$

for $1 \le n \le 2j-1$.

Proof. The dimension of W^n_j is $(2n-1)^2 - (n)^2 = 3n^2 - 4n + 1$. It is easy to count that there are $(2n-1)^2 - (n)^2 = 3n^2 - 4n + 1$ functions in the following set

$$\begin{array}{ll} \psi^{j,1}_{0,k}, & k = 1, .., n; \\ \psi^{j,2}_{k,0}, & k = 1, .., n; \\ \psi^{j,3}_{i,k}, & 1 \le i, k \le n; \\ \psi^{j,4}_{i,k}, & 1 \le i, k \le n; \\ \psi^{j,5}_{i,k}, & 1 \le i, k \le n \end{array}$$

which all belong to the space W^n_j. Since they are linear independent, they form a basis for space W^n_j, where $1 \le n \le 2j-1$. □

Finally we need to find wavelets in $W_j^{2^j} \backslash W_j^{2^j-1}$. The computations are the same to the above except for that there is one globally supported basis function. In fact the following pictures show the basis functions located on the top boundary of the domain Ω. (We omit the pictures for the basis functions on the right vertical boundary which are symmetric with respect to the line y=x are those basic functions on the top horizontal boundary of Ω.)

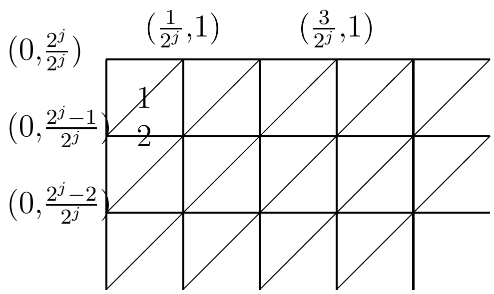

Figure 19.

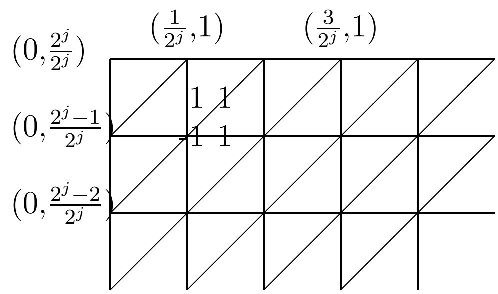

Figure 20.

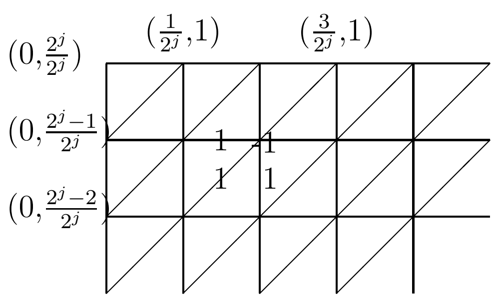

Figure 21.

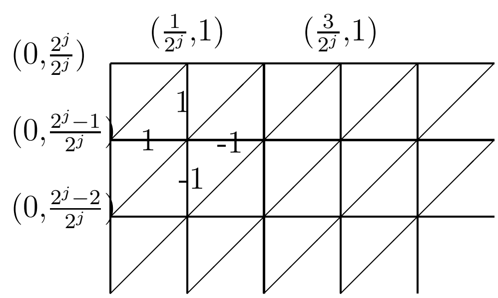

Figure 22.

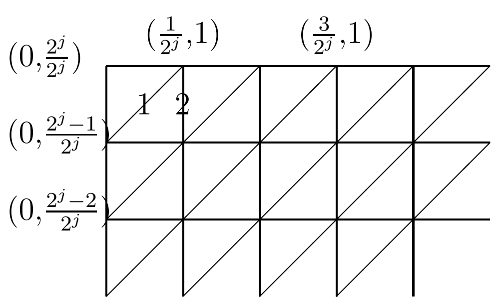

Figure 23.

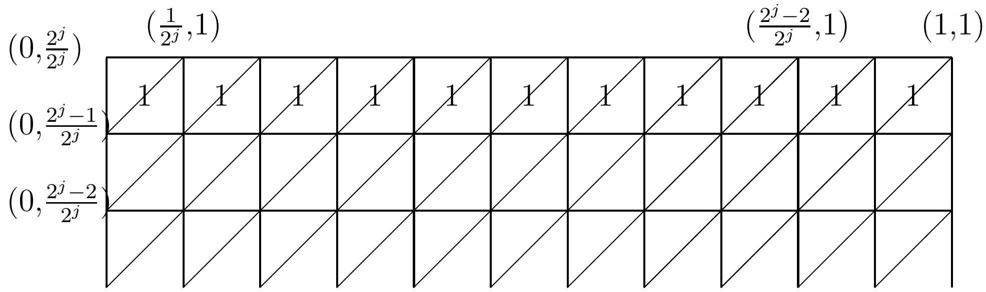

Figure 24.

The last one (cf. Figure 24) is the only special basis function since it is not local supported. The numbers of all these wavelets in $W_j^{2^j}\backslash W_j^{2^j-1}$ amount to $2^{j+3}-8$ which is equal to the number of dimension of $V_{j+1}^{2^j}\backslash V_{j+1}^{2^j-1}$.

Theorem 3.3. *All the wavelets in the $W_j^{2^j}\backslash W_j^{2^j-1}$ are linear independent and form a basis for $V_{j+1}^{2^j}\backslash V_{j+1}^{2^j-1}$ which is spanned by the functions in $\{\phi_{i,k}^{j+1}, 2^{j+1}-2 \le i,k \le 2^{j+1}-1\}$.*

Proof. Let us just concentrate on the basis functions in $V_{j+1}^{2^j}\backslash V_{j+1}^{2^j-1}$ and in $W_j^{2^j}\backslash W_j^{2^j-1}$. Then the scaling matrix between two sets of basis functions is the following matrix up to a constant

$$A = \left(\begin{array}{ccccccccccc}
D & & & & & & & & & & \\
B1 & B2 & & & & & & & & & \\
 & B1 & B2 & & & & & & & & \\
 & & B1 & B2 & & & & & & & \\
 & & & \ddots & & & & & & & \\
 & & & & B1 & B2 & & & & & \\
 & & & & & C1 & C2 & & & & \\
C3 & C3 & C3 & \dots & C3 & C3 & C3 & & & & \\
 & & & & & & C4 & & & & \\
 & & & & & & B2' & B1' & & & \\
 & & & & & & & B2' & B1' & & \\
 & & & & & & & & \ddots & & \\
 & & & & & & & & & B2' & B1' \\
 & & & & & & & & & & D'
\end{array}\right),$$

where

$$D = \left(\begin{array}{cccc} 1 & 2 & 0 & 0 \end{array}\right), \quad B1 = \left(\begin{array}{cccc} 1 & 0 & 2 & \\ & 1 & 0 & -1 \\ & 1 & 1 & 0 \\ & & 1 & -1 \end{array}\right), \quad B2 = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right),$$

$$D' = \begin{pmatrix} 0 & 0 & 2 & 1 \end{pmatrix}, \quad B1' = \begin{pmatrix} -1 & 1 & & \\ 0 & 1 & 1 & \\ -1 & 0 & 1 & \\ & 2 & 0 & 1 \end{pmatrix}, \quad B2' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C1 = \begin{pmatrix} 1 & 0 & 2 & \\ & 1 & 0 & -1 \\ & 1 & 1 & 0 \\ & & 1 & -1 \end{pmatrix}, \quad C2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad C4 = \begin{pmatrix} 0 & 2 & 0 & 1 \end{pmatrix},$$

$$C3 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $E = (m\ n\ 0\ 0)$. By the row operations we have

$$\begin{pmatrix} E & & \\ B1 & B2 & \\ & B1 & B2 \end{pmatrix} = \begin{pmatrix} m & n & 0 & & & & & & & \\ 1 & 0 & 2 & & & & & & & \\ & 1 & 0 & -1 & & & & & & \\ & 1 & 1 & 0 & 0 & -1 & & & & \\ & & 1 & -1 & 1 & 1 & & & & \\ & & & & 1 & 0 & 2 & & & \\ & & & & & 1 & 0 & -1 & & \\ & & & & & 1 & 1 & 0 & 0 & -1 \\ & & & & & & 1 & -1 & 1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} m & n & & & & & & & & \\ & -n & 2m & & & & & & & \\ & & 2m & -n & & & & & & \\ & & & n & m & & & & & \\ & & & & 2m+n & 2n & 0 & 0 & & \\ & & & & 1 & 0 & 2 & & & \\ & & & & & 1 & 0 & -1 & & \\ & & & & & 1 & 1 & 0 & 0 & -1 \\ & & & & & & 1 & -1 & 1 & 1 \end{pmatrix}.$$

Similar for B'. Thus by row operations,

$$A \to \begin{pmatrix} A_1 & G_1 & & & & & & & & & \\ & A_2 & G_2 & & & & & & & & \\ & & A_3 & G_3 & & & & & & & \\ & & & \ddots & & & & & & & \\ & & & & A_{2^j-2} & G_{2^j-2} & & & & & \\ & & & & & C_1' & C_2' & & & & \\ & & & & & & G'_{2^j-2} & A'_{2^j-2} & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & G_2' & A_2' & \\ & & & & & & & & & G_1' & A_1' \end{pmatrix},$$

where A_n is an upper triangular matrix of size 4×4 while A'_n is a lower triangular matrix of size 4×4 which are given below.

$$A_1 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ & -1 & 1 & 0 \\ & & 1 & -1 \\ & & & 2 \end{pmatrix}, \; G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ & -1 & 2 & 0 \\ & & 2 & -1 \\ & & & 1 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; A_n = \begin{pmatrix} n & 2 & 0 & 0 \\ & -1 & n & 0 \\ & & n & -1 \\ & & & 2 \end{pmatrix}, \; G_n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ n & 0 & 0 & 0 \end{pmatrix},$$

$$A'_n = \begin{pmatrix} 2 & & & \\ -1 & n & & \\ 0 & n & -1 & \\ 0 & 0 & 2 & n \end{pmatrix}, \; G'_n = \begin{pmatrix} 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the matrix $(C'_1 \quad C'_2)$ is the following matrix

$$(C'_1 \quad C'_2) = \begin{pmatrix} 2^{j+1}-5 & 2 & & & & & & \\ 1 & 0 & 2 & & & & & \\ & 1 & 0 & -1 & 0 & 0 & 1 & \\ & 1 & 1 & 0 & 0 & -1 & -1 & \\ & & 1 & -1 & 1 & 1 & & \\ 2^{j+1}-5 & 0 & 0 & 0 & 1 & 0 & & \\ & & & & & 2 & 0 & 1 \\ & & & & & & 2 & 2^{j+1}-5 \end{pmatrix}.$$

It is easy to see the rank of $(C'_1 \quad C'_2)$ is 8. Thus the rank of A is $8(2^j) - 8$. Thus, all the prewavelet functions constructed above in the $W_j^{2^j} \backslash W_j^{2^j-1}$ are linear independent and hence form a basis of $V_{j+1}^{2^j} \backslash V_{j+1}^{2^j-1}$. $\square$

It is easy to see that the coefficients of the prewavelet functions in $W_j^{2^j-1}$ in terms of the basis functions of $V_{j+1}^{2^j} \backslash V_{j+1}^{2^j-1}$ are all zeros. Thus the prewavelet functions in $W_j^{2^j-1}$ together with the prewavelet functions in $V_{j+1}^{2^j} \backslash V_{j+1}^{2^j-1}$ are linear independent. It follows the main result in this paper.

Theorem 3.4. *All the prewavelet functions in the $W_j^{2^j} \backslash W_j^{2^j-1}$ and the prewavelet functions in $W_j^{2^j-1}$ form a basis for W_j.*

4 The Prewavelet Method for Poisson Equation

Let us use the basis functions of V_j and W_j to solve Poisson equation (1). Mainly we explain how to compute $h_j \in W_j$. Let $g_j \in V_j$ and $g_{j+1} \in V_{j+1}$ be two FEM solutions. We aim to show that $h_j + g_j = g_{j+1}$.

By a reordering the indices $(i,k), 1 \leq i,k \leq 2^j$ in a linear fashion, let $V_j = span\{\phi_1^j, ..., \phi_{N_j}^j\}$. Also, we reorder all five type wavelet functions as well as the globally supported wavelet to denote $W_j = span\{\psi_1^j, ...\psi_{N_{j+1}-N_j}^j\}$. Let Φ^j, Ψ^j be following vectors,

$$\Phi^j = \begin{pmatrix} \phi_1^j \\ \phi_2^j \\ \vdots \\ \phi_{N_j}^j \end{pmatrix}, \qquad \Psi^j = \begin{pmatrix} \psi_1^j \\ \psi_2^j \\ \vdots \\ \psi_{N_{j+1}-N_j}^j \end{pmatrix}.$$

Then we have the following equations

$$\Phi^j = B_j \Phi^{j+1}, \qquad \Psi^j = C_j \Phi^{j+1},$$

where B_j is $N_j \times N_{j+1}$ refinable matrix, and C_j is a wavelet matrix of size $(N_{j+1} - N_j) \times N_{j+1}$. Let D_j and E_j be the following matrices:

$$D_j = \begin{pmatrix} \langle \phi_1^j, \phi_1^j \rangle_s & \langle \phi_1^j, \phi_2^j \rangle_s & \cdots\cdots & \langle \phi_1^j, \phi_{N_j}^j \rangle_s \\ \langle \phi_2^j, \phi_1^j \rangle_s & \langle \phi_2^j, \phi_2^j \rangle_s & \cdots\cdots & \langle \phi_2^j, \phi_{N_j}^j \rangle_s \\ \vdots & \vdots & \ddots & \vdots \\ \langle \phi_{N_j}^j, \phi_1^j \rangle_s & \langle \phi_{N_j}^j, \phi_2^j \rangle_s & \cdots\cdots & \langle \phi_{N_j}^j, \phi_{N_j}^j \rangle_s \end{pmatrix}$$

$$E_j = \begin{pmatrix} \langle \psi_1^j, \psi_1^j \rangle_s & \langle \psi_1^j, \psi_2^j \rangle_s & \cdots\cdots & \langle \psi_1^j, \psi_{N_{j+1}-N_j}^j \rangle_s \\ \langle \psi_2^j, \psi_1^j \rangle_s & \langle \psi_2^j, \psi_2^j \rangle_s & \cdots\cdots & \langle \psi_2^j, \psi_{N_{j+1}-N_j}^j \rangle_s \\ \vdots & \vdots & \ddots & \vdots \\ \langle \psi_{N_{j+1}-N_j}^j, \psi_1^j \rangle_s & \langle \psi_{N_{j+1}-N_j}^j, \psi_2^j \rangle_s & \cdots\cdots & \langle \psi_{N_{j+1}-N_j}^j, \psi_{N_{j+1}-N_j}^j \rangle_s. \end{pmatrix}.$$

It is easy to see that $B_j D_{j+1} C_j^T = 0$ is equivalent to $V_j \perp W_j$. Clearly, we have $D_j = B_j D_{j+1} B_j^T$ and $E_j = C_j D_{j+1} C_j^T$.

Let g_j be the projection of g in V_j, and h_j be the projection of g in W_j. Since $V_j \bigoplus W_j = V_{j+1}$, $g_j + h_j$ will be equal to g_{j+1}. Let us write $g_j = \sum_{j=1}^{N_j} a_i \phi_i^j = (a_1, a_2, ..., a_{N_j})\Phi^j$. Similarly, $h_j = (b_1, b_2, ..., b_{N_{j+1}-N_j})\Psi^j$, and $g_{j+1} = (c_1, c_2, ..., c_{N_{j+1}})\Phi^{j+1}$. By computing the weak solutions h_j, g_j, and g_{j+1} in W_j, V_j, and V_{j+1}, respectively, we have

$$D_j \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{N_j} \end{pmatrix} = \begin{pmatrix} \langle \phi_1^j, g \rangle \\ \langle \phi_2^j, g \rangle \\ \vdots \\ \langle \phi_{N_j}^j, g \rangle \end{pmatrix},$$

$$E_j \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{N_{j+1}-N_j} \end{pmatrix} = \begin{pmatrix} \langle \psi_1^j, g\rangle \\ \langle \psi_2^j, g\rangle \\ \vdots \\ \langle \psi_{N_{j+1}-N_j}^j, g\rangle \end{pmatrix},$$

$$D_{j+1} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N_{j+1}} \end{pmatrix} = \begin{pmatrix} \langle \phi_1^{j+1}, g\rangle \\ \langle \phi_2^{j+1}, g\rangle \\ \vdots \\ \langle \phi_{N_{j+1}}^{j+1}, g\rangle \end{pmatrix}.$$

It follows

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{N_j} \end{pmatrix} = (D_j)^{-1} \begin{pmatrix} \langle \phi_1^j, g\rangle \\ \langle \phi_2^j, g\rangle \\ \vdots \\ \langle \phi_{N_j}^j, g\rangle \end{pmatrix},$$

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{N_{j+1}-N_j} \end{pmatrix} = (E_j)^{-1} \begin{pmatrix} \langle \psi_1^j, g\rangle \\ \langle \psi_2^j, g\rangle \\ \vdots \\ \langle \psi_{N_{j+1}-N_j}^j, g\rangle \end{pmatrix},$$

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N_{j+1}} \end{pmatrix} = (D_{j+1})^{-1} \begin{pmatrix} \langle \phi_1^{j+1}, g\rangle \\ \langle \phi_2^{j+1}, g\rangle \\ \vdots \\ \langle \phi_{N_{j+1}}^{j+1}, g\rangle \end{pmatrix}.$$

The above linear systems provide a computational method to find g_j, h_j.

We now show $h_j + g_j = g_{j+1}$. That is, g_{j+1} can be computed by using h_j and g_j only. Indeed, we have

$$\begin{aligned} g_j &= (a_1, a_2, ..., a_{N_j})\Phi^j = (\Phi^j)^T (a_1, a_2, ..., a_{n_j})^T \\ &= (\Phi^{j+1})^T B_j^T (a_1, a_2, ..., a_{N_j})^T \\ &= (\Phi^{j+1})^T B_j^T D_j^{-1} (\langle \phi_1^j, g\rangle, \langle \phi_2^j, g\rangle, \cdots \langle \phi_{N_j}^j, g\rangle)^T \\ &= ((\Phi^{j+1}))^T B_j^T (B_j D_{j+1} B_j^T)^{-1} B_j (\langle \phi_1^{j+1}, g\rangle, \langle \phi_2^{j+1}, g\rangle, \cdots \langle \phi_{N_{j+1}}^{j+1}, g\rangle)^T. \end{aligned}$$

Similarly,

$$h_j \;\; = \;\; ((\Phi^{j+1}))^T C_j^T (C_j D_{j+1} C_j^T)^{-1} C_j (\langle \phi_1^{j+1}, g\rangle, \langle \phi_2^{j+1}, g\rangle, \cdots \langle \phi_{N_{j+1}}^{j+1}, g\rangle)^T.$$

and

$$g_{j+1} \;\; = \;\; ((\Phi^{j+1}))^T D_{j+1}^{-1} (\langle \phi_1^{j+1}, g\rangle, \langle \phi_2^{j+1}, g\rangle, \cdots \langle \phi_{N_{j+1}}^{j+1}, g\rangle)^T.$$

In order to show $h_j + g_j = g_{j+1}$, we only need to prove

$$B_j^T (B_j D_{j+1} B_j^T)^{-1} B_j + C_j^T (C_j D_{j+1} C_j^T)^{-1} C_j = D_{j+1}^{-1}. \tag{14}$$

Notice that B_j and C_j are not square matrices. That is we can not invert B_j and C_j. Consider

$$\begin{aligned} \begin{pmatrix} B_j \\ C_j \end{pmatrix} D_{j+1} \begin{pmatrix} B_j^T & C_j^T \end{pmatrix} &= \begin{pmatrix} B_j D_{j+1} B_j^T & B_j D_{j+1} C_j^T \\ C_j D_{j+1} B_j^T & C_j D_{j+1} C_j^T \end{pmatrix} \\ &= \begin{pmatrix} B_j D_{j+1} B_j^T & 0 \\ 0 & C_j D_{j+1} C_j^T \end{pmatrix} \end{aligned}$$

by using the orthogonal conditions of V_j and W_j. Then we have the following equation

$$\begin{pmatrix} B_j D_{j+1} \\ C_j D_{j+1} \end{pmatrix} \begin{pmatrix} B_j^T & C_j^T \end{pmatrix} \begin{pmatrix} (B_j D_{j+1} B_j^T)^{-1} & 0 \\ 0 & (C_j D_{j+1} C_j^T)^{-1} \end{pmatrix} = I,$$

where I stands for the identity matrix. In other words, we have

$$\begin{pmatrix} B_j D_{j+1} \\ C_j D_{j+1} \end{pmatrix} \begin{pmatrix} B_j^T (B_j D_{j+1} B_j^T)^{-1} & C_j^T (C_j D_{j+1} C_j^T)^{-1} \end{pmatrix} = I$$

which can be rewritten in the following form

$$\begin{pmatrix} B_j^T (B_j D_{j+1} B_j^T)^{-1} & C_j^T (C_j D_{j+1} C_j^T)^{-1} \end{pmatrix} \begin{pmatrix} B_j D_{j+1} \\ C_j D_{j+1} \end{pmatrix} = I.$$

Hence we have

$$B_j^T (B_j D_{j+1} B_j^T)^{-1} B_j D_{j+1} + C_j^T (C_j D_{j+1} C_j^T)^{-1} C_j D_{j+1} = I$$

or

$$B_j^T (B_j D_{j+1} B_j^T)^{-1} B_j + C_j^T (C_j D_{j+1} C_j^T)^{-1} C_j = D_{j+1}^{-1}.$$

which is (14) and hence $h_j + g_j = g_{j+1}$.

5 Numerical Experiments

We have implemented the prewavelet method for numerical solution of Poisson equations over rectangular domains in MATLAB. We would like to demonstrate that our prewavelet method is more efficient than the standard FEM method.

In the following we provide three tables of CPU times for numerical solutions based on our prewavelet method and the standard finite element method for various levels of refinement of an initial triangulation (Γ_0 which consists of two triangles) of the standard domain $[0, 1] \times [0, 1]$.

Let V_j be the continuous linear finite element space over triangulation Γ_j which is the jth refinement of Γ_0. For a test function u which is the exact solution of Poisson equation (1), the finite element method is to compute $u_j \in V_j$ directly while our prewavelet method computes u_j by computing $w_k, k = 1, \cdots, j$, i.e., $u_j = u_1 + w_1 + \cdots + w_{j-1}$.

In the following we present three tables of CPU times for computing numerical solutions $u_j, j = 4, 5, 6$ for three test solutions by using these two methods. Note that we use the direct method coded in MATLAB to solve the associated linear equations. We shall present tables of CPU times based on Conjugate Gradient Method for the systems of equations next.

For an exact solution $u(x, y) = \sin(2\pi x)\sin(2\pi y)$ which clearly satisfies the zero boundary conditions, we list CPU times for computing numerical solutions $u_j, j = 4, 5, 6$ by using these two methods in Table 1.

Table 1. CPU times to compute u_j by the two methods

	FEM method	Prewavelet Method
j=4	0.164531 seconds	0.204067 seconds
j=5	0.593587 seconds	0.519293 seconds
j=6	13.960323 seconds	6.222679 seconds

For an exact solution $u(x, y) = xy(1 - x)(1 - y)$, the CPU times for numerical solutions by these two methods are given in Table 2.

Table 2. CPU times for computing u_j by the two methods

CPU time	FEM method	Prewavelet Method
j=4	0.150836 seconds	0.218282 seconds
j=5	0.574085 seconds	0.558071 seconds
j=6	13.896825 seconds	6.202557 seconds

We list the CPU times for computing numerical solutions $u_j, j = 4, 5, 6$ of $u(x, y) = xy(1 - x)(1 - y)e^{8xy}$ by using these two methods in Table 3.

Table 3. CPU times for computing u_j by the two methods

CPU time	FEM method	Prewavelet Method
j=4	0.144159 seconds	0.186389 seconds
j=5	0.584828 seconds	0.459181 seconds
j=6	13.877403 seconds	6.139101 seconds

It is clear from these three tables that the prewavelet method is much more efficient.

Next we use the Conjugate Gradient Method to solve the linear systems associated with FEM. Let us consider iterative solution to u_j for $j = 6$ with various accuracy. First let us consider the exact solution $u(x, y) = \sin(2\pi x)\sin(2\pi y)$.

Table 4. CPU times for approximating the FEM solution u_6 by Conjugate Gradient Method

ϵ	CPU times
10^{-8}	5.411852 seconds
10^{-9}	5.783497 seconds
10^{-10}	6.221683 seconds
10^{-11}	6.616816 seconds
10^{-12}	6.917468 seconds
10^{-13}	7.836775 seconds

To approximate the FEM solution u_6 of the exact solution $u(x, y) = xy(1-x)(1-y)$ by the Conjugate Gradient Method, we list the CPU times in Table 5.

Table 5. CPU times for approximating the FEM solution u_j by Conjugate Gradient Method

ϵ	CPU times
10^{-8}	4.476794 seconds
10^{-9}	4.878259 seconds
10^{-10}	5.306747 seconds
10^{-11}	5.887849 seconds
10^{-12}	6.811317 seconds
10^{-13}	6.754465 seconds

Finally let us consider the CPU times to approximate the FEM solution u_6 of $u(x, y) = xy(1-x)(1-y)e^{8xy}$ by the Conjugate Gradient Method.

Table 6. CPU times for approximating the FEM solution by Conjugate Gradient Method

ϵ	CPU times
10^{-8}	10.110517 seconds
10^{-9}	10.740035 seconds
10^{-10}	11.319618 seconds
10^{-11}	11.810142 seconds
10^{-12}	12.320903 seconds
10^{-13}	13.103407 seconds

It is clear from all six tables, if we want an accurate iterative solution of u_6 within 10^{-12}, the prewavelet method appears better.

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